AN EMPIRICAL STUDY OF THE STABILITIES OF ESTIMATORS AND VARIANCE ESTIMATORS IN p.p.s. SAMPLING

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1. Introduction

Several methods of probability proportional to size (p.p.s.) sampling without replacement have been proposed in recent years. However, not much is known with regard to the stabilities of estimators of the population total, and practically nothing is known with regard to the stabilities of their variance estimators. Therefore, in the present paper, we make an empirical study of the stabilities of estimators and variance estimators for the important case of sample size n=2, using several natural as well as artificial populations. The artificial populations are deliberately chosen to represent situations more extreme than those normally encountered in practice and they provide better discrimination among the methods.

We have chosen only those methods (excepting one) which satisfy the following essential requirements: (a) Variance of the estimator should be smaller than that of the customary estimator in p.p.s. sampling with replacement. (b) A non-negative, unbiased variance estimator should be available. (c) Computations involved should not be cumbersome. Requirement (b) eliminates the systematic method of Madow (1949) and Hartley (1966). We have not included the asymptotic methods (valid for large or moderate population size N) of Hartley and Rao (1962), Rao (1963) and Hajek (1964), although they satisfy the above requirements. Similarly, Stevens' (1958) method is excluded as it is applicable only when the units in the population are or can be grouped with respect to the sizes x_j such that units in a group have the same size.

Based on the above considerations, we have selected the following methods for the present study: (1) The methods of Brewer (1963), Carroll and Hartley (1964), Fellegi (1963), Rao (1963, 65), Durbin (1967), and Hanurav (1967), all using the Horvitz-Thompson estimator and satisfying $\pi_j=2x_j/X$ where π_j is the probability of selecting the j^{th} population unit ($j=1,2,\ldots,N$) in the sample and $X=\Sigma x_j$, (2) the methods of Des Raj (1956) and Murthy (1957), (3) the method of Rao, Hartley and Cochran (1962) and (4) Lahiri's (1951) method using a ratio estimator.

The methods of Brewer, Durbin and Rao (1965) are equivalent in that they have the same joint probabilities of selection π_{ij} and, hence, the same variance. However, Durbin's method has the advantage of allowing for rotation of the sample. For convenience, we denote this group of methods as Brewer's method. The methods of Rao (1963) (investigated in detail by Carroll and Hartley) and Fellegi are equivalent and both possess rotational properties. Again for convenience we denote this group as Fellegi's method. Hanurav has proposed the criterion $\varphi = \min \pi_{ij}/(\pi_i\pi_j) > \beta$, for β sufficiently away from zero to improve the stability of the variance estimator and developed a method which satisfies this criterion except when the largest size $\tilde{x}_N$ is markedly different from the next largest size $\tilde{x}_{N-1}$. He has, however, not shown whether his method satisfies requirement (a), although it appears highly probable. The methods of Brewer and Fellegi also seem to satisfy Hanurav's criterion, except when $\tilde{x}_N/X$ is very close to 1/2. For instance, if $\tilde{x}_N/X \le 1/3$ and the other $x_i/X \le 1/4$, $\varphi > 0.3$ for Brewer's method; the above bound, however, is conservative and the actual value of φ is normally much larger.

The method of Des Raj depends on the order in which the units are drawn and it is known that Murthy's estimator is uniformly more efficient than that of Des Raj. The requirements (a) and (b) are not satisfied by Lahiri's estimator. Nevertheless we have included it in veiw of the recent work by Godambe (1966) based on concepts other than efficiency.

The methods in (1) have an advantage over the others in that the estimates become self-weighting with equal work loads within the selected primaries whereas the others require random work loads. We shall, however, not consider this as a factor in the choice among the methods.

The computations involved in applying the methods of Brewer, Murthy, Des Raj, Lahiri or Rao, Hartley and Cochran (R.H.C.) are very simple and about the same amount. Hanurav's method is slightly more involved whereas Fellegi's method involves simple, iterative calculations. In any case, the choice among these methods based on computational simplicity is not very realistic, especially when a high-speed computer is available.

We supplement our empirical study with a semi-theoretical study based on a super-population approach in which the finite population is regarded as being drawn from an infinite super-population. The results obtained apply only to the average of all finite populations that can be drawn from the super-population. We assume the following, often used, super-population model for the comparison of estimators:

$$y_i = \beta x_i + e_i,\ i=1,\ldots,N$$
$$\epsilon(e_i|x_i) = 0,\ \epsilon(e_i^2|x_i) = ax_i^g \quad (1)$$
$$\epsilon(e_ie_j|x_i,x_j) = 0,\ a > 0,\ g \geq 0$$

where ϵ denotes the average over all the finite populations that can be drawn from the super-population. For the comparison of variance estimators we further assume that e_i's are normally distributed. In most practical situations, $1 \leq g \leq 2$. Some theoretical results are available on the relative efficiencies of the estimators (Rao, 1966) under the above model, but no guidelines are available with regard to the relative magnitudes. Nothing is known on the stabilities of the variance estimators under the super-population model.

2. Formulae

Let Y denote the population total of the characteristics of interest $y_i (i=1,2,\ldots,N)$. For the methods in group (1), the Horvitz-Thompson estimator of Y is

$$\hat{Y}_1 = \frac{1}{2}\left(\frac{y_1}{p_1} + \frac{y_2}{p_2}\right) \quad (2)$$

with variance

$$V_1 = \sum_{i<j}^{N}\sum \left(p_ip_j - \frac{\pi_{ij}}{4}\right)\left(\frac{y_i}{p_i} - \frac{y_j}{p_j}\right)^2 \quad (3)$$

and variance estimator (Yates-Grundy)

$$v_1 = \frac{4p_1p_2-\pi_{12}}{4\pi_{12}}\left(\frac{y_1}{p_1} - \frac{y_2}{p_2}\right)^2 \quad (4)$$

where 1 and 2 denote the two units in the sample and $p_i = x_i/X$. The variance of the variance estimator is $Ev_1^2 - V_1^2$ where

$$16[Ev_1^2] = \sum_{i<j}^{N}\sum \frac{(4p_ip_j-\pi_{ij})^2}{\pi_{ij}}\left(\frac{y_i}{p_i} - \frac{y_j}{p_j}\right)^4. \quad (5)$$

The formulae for π_{ij} for the various methods can be obtained from Durbin (1967), Fellegi (1963) and Hanurav (1967).

For Des Raj's method, the estimator of Y is

$$\hat{Y}_2' = \frac{1}{2}\left[y_1'\frac{(1+p_1')}{p_1'} + y_2'\frac{(1-p_1')}{p_2'}\right] \quad (6)$$

with variance

$$V_2' = \frac{1}{4}\sum_{i<j}^{N}\sum p_ip(2-p_i-p_j)\left(\frac{y_i}{p_i} - \frac{y_j}{p_j}\right)^2 \quad (7)$$

and variance estimator

$$v_2' = \frac{(1-p_1')^2}{4}\left(\frac{y_1}{p_1} - \frac{y_2}{p_2}\right)^2 \quad (8)$$

where (y_1',p_1') and (y_2',p_2') denote the y- and p-values of the units selected at the first and second draws respectively. The variance of the variance estimator is $Ev_2'^2 - V_2'^2$ where

$$16[Ev_2'^2] = \sum_{i=0}^{3}\sum_{j=0}^{4}(-1)^{1-i-j}\binom{3}{i}\binom{4}{j}\cdot A_{j,i+j-4}\,A_{4-j,3-j} \quad (9)$$

where

$$A_{ij} = \sum_{t=1}^{N}\frac{y_t^i}{p_t^j}. \quad (10)$$

Murthy's estimator of Y is

$$\hat{Y}_3 = \frac{1}{2-p_1-p_2}\left[(1-p_2)\frac{y_1}{p_1} + (1-p_1)\frac{y_2}{p_2}\right] \quad (11)$$

with variance

$$V_3 = \sum_{i<j}^{N}\sum p_ip_j\frac{(1-p_i-p_j)}{(2-p_i-p_j)}\left(\frac{y_i}{p_i} - \frac{y_j}{p_j}\right)^2 \quad (12)$$

and variance estimator

$$v_3 = \frac{(1-p_1)(1-p_2)(1-p_1-p_2)}{(2-p_1-p_2)^2}\left(\frac{y_1}{p_2} - \frac{y_2}{p_2}\right)^2. \quad (13)$$

Also

$$Ev_3^2 = \sum_{i<j}^{N}\sum p_ip_j\frac{(1-p_i)(1-p_j)(1-p_i-p_j)^2}{(2-p_i-p_j)^3}\cdot\left(\frac{y_i}{p_i} - \frac{y_j}{p_j}\right)^4. \quad (14)$$

In the Rao-Hartley-Cochran method, the population is split at random into 2 groups of sizes N_1 and N_2 $(N_1+N_2=N)$ and a sample of size one is drawn with probabilities proportional to p_t from each of the two groups independently. Their estimator of Y is

$$\hat{Y}_4 = \frac{y_1}{p_1}P_1 + \frac{y_2}{p_2}P_2 \quad (15)$$

where $P_i = \sum_{\text{group } i} p_t$, $(i=1,2)$. The variance and variance estimator of $\hat{Y}_4$ are given by

$$V_4 = 2c_0c_1(A_{21} - A_{10}^2) \tag{16}$$

and

$$v_4 = c_0 \sum_1^2 P_i\left(\frac{y_i}{p_i} - \hat{Y}_4\right)^2$$

$$= c_0P_1P_2\left(\frac{y_1}{p_1} - \frac{y_2}{p_2}\right)^2 \tag{17}$$

respectively, where the A_{ij} are given by (10) and

$$c_0 = \frac{N_1^2+N_2^2-N}{N^2-N_1^2-N_2^2}, \quad c_1 = \frac{N^2-N_1^2-N_2^2}{N(N-1)} . \tag{18}$$

The derivation of Ev_4^2 is very tedious but straight forward. We have

$$\begin{aligned} c_0^{-2}[Ev_4^2] = {} & c_1[4A_{30}A_{10}-4A_{10}A_{31}+3A_{20}^2-6A_{40}] \\ & + c_2[A_{41}+2A_{42}A_{0,-2}-A_{42}+12A_{40} \\ & - 6A_{21}A_{20}+6A_{21}A_{2,-1}+3A_{21}^2-3A_{21}^2A_{0,-2} \\ & + A_{43}A_{0,-3}-12A_{30}A_{10}+12A_{31}A_{10} \\ & + 4A_{10}A_{32}A_{0,-2}-4A_{32}A_{1,-2}-8A_{31}A_{1,-1} \\ & + A_{43}-2A_{43}A_{0,-2}-4A_{32}A_{10}+4A_{32}A_{1,-1}] \\ & +c_3[2A_{42}-2A_{41}-A_{42}A_{0,-2}-6A_{20}^2 \\ & + 12A_{21}A_{20}-12A_{21}A_{2,-1}-3A_{21}^2 \\ & + 3A_{21}^2A_{0,-2}-2A_{43}A_{0,-3}+3A_{43}A_{0,-2} \\ & + 8A_{30}A_{10}-8A_{31}A_{10}+4A_{32}A_{10} \\ & - 4A_{32}A_{10}A_{0,-2}+8A_{31}A_{1,-1} \\ & - 8A_{32}A_{1,-1}-A_{43}+8A_{32}A_{1,-2}] \end{aligned} \tag{19}$$

where

$$c_2 = \frac{N_1N_2(N_1+N_2-2)}{N(N-1)(N-2)}, \quad c_3 = \frac{N_1N_2(N_1^2+N_2^2-3N+4)}{N(N-1)(N-2)(N-3)}. \tag{20}$$

Lahiri's estimator of Y is

$$\hat{Y}_5 = (y_1+y_2)/(p_1+p_2) \tag{21}$$

with variance

$$V_5 = \frac{1}{N-1} \sum_{i<j}^{N}\sum \frac{(y_i+y_j)^2}{p_i+p_j} - Y^2 \tag{22}$$

and variance estimator

$$v_5 = \hat{Y}_5^2 - \frac{1}{p_1+p_2}\left[(y_1-y_2)^2+2Ny_1y_2\right] \tag{23}$$

which takes negative values. Further

$$Ev_5^2 = \frac{1}{N-1} \sum_{i<j}^{N}\sum(p_i+p_j)\left[\hat{Y}_5^2 - \frac{1}{p_i+p_j}\left\{(y_i-y_j)^2 + 2Ny_iy_j\right\}\right]^2. \tag{24}$$

The variance of the customary estimator in p.p.s. sampling with replacement is given by

$$V_6 = (A_{21}-A_{10}^2)/2. \tag{25}$$

The variance estimator is

$$v_6 = \frac{1}{4}\left(\frac{y_1}{p_1} - \frac{y_2}{p_2}\right)^2 \tag{26}$$

and

$$8[Ev_6^2] = A_{43}+3A_{21}^2-4A_{32}A_{10}. \tag{27}$$

In (18) and (20) we have taken N_i=N/2 when N is even and N_1=(N-1)/2 and N_2=(N+1)/2 when N is odd.

3. Empirical Results

We have chosen 7 artificial and 20 natural populations for the empirical study. Table 1 gives the source, nature of y and x, coefficients of variation (C.V.) of y and x, correlation ρ and the ratio φ for the methods of Brewer, Fellegi and Hanurav. It is clear from Table 1 that we have a wide variety of populations with N ranging from 4 to 35, and C.V.(x) from 0.14 to 1.26 and ρ from 0.59 to 0.999. For the natural populations, Hanurav's criterion is satisfied by the methods of Brewer, Fellegi and Hanurav, except possibly for population 5 where $\tilde{x}_N$ is markedly different from $\tilde{x}_{N-1}$ and close to X/2. In general, Hanurav's ratio appears slightly larger than Fellegi's which in turn is slightly larger than Brewer's ratio. Turning to artificial populations, we see that all the three ratios are very close to zero for population 7 in which $\tilde{p}_N$=0.49 and $\tilde{p}_{N-1}$=0.30. For population 6, we took $\tilde{p}_N$=0.42 and $\tilde{p}_{N-1}$=0.40 so that Hanurav's ratio is considerably larger than Brewer's ratio. In any case, it is clear from these examples that none of the three methods guarantee that φ will be sufficiently away from zero for all populations. It also appears that, under Hanurav's criterion, the stabilities of the variance estimators should be about equal for these three methods. We shall, however, provide direct evidence on this point by computing the coefficients of variation of the variance estimators.

Table 1. Description of the populations

No	Source	y	x	N	C.V.(y)	C.V.(x)	ρ	φ Fellegi	φ Brewer	φ Hanurav
Artificial Populations										
1	Cochran (1963)	Artificial	Artificial	5	0.57	0.50	0.87	0.44	0.41	0.52
2	Cochran (1963)	Artificial	Artificial	5	0.68	0.50	0.997	0.44	0.41	0.52
3	Yates & Grundy (1953)	Artificial	Artificial	4	0.67	0.52	0.995	0.39	0.35	0.40
4	Yates & Grundy (1953)	Artificial	Artificial	4	0.50	0.52	0.88	0.39	0.35	0.40
5	Fellegi (1963)	Artificial	Artificial	6	0.64	0.25	0.93	0.52	0.52	0.53
6	Present Authors	Artificial	Artificial	4	0.72	0.74	0.999	0.33	0.21	0.46
7	Present Authors	Artificial	Artificial	4	0.78	0.80	0.997	0.07	0.05	0.06
Natural Populations										
1	Horvitz & Thompson (1952)	No. of households	Eye-estimated no.of households	20	0.44	0.40	0.87	0.49	0.49	0.50
2	DesRaj (1965) (Modification of 1)	No. of households	Eye-estimated no.of households	20	0.44	0.41	0.66	0.49	0.49	0.50
3	Rao (1963)	Corn acreage in 1960	Corn acreage in 1958	14	0.39	0.43	0.93	0.49	0.49	0.50
4	Kish (1965) blocks 1-10	No.of rented dwelling units	Total no.of dwelling units	10	1.45	1.15	0.99	0.36	0.32	0.42
5	Kish (1965) blocks 11-20	No.of rented dwelling units	Total no.of dwelling units	10	1.25	1.26	0.98	0.26	0.21	0.23
6	Cochran (1963)	Wt. of peaches	eye-estimated wt. of peaches	10	0.19	0.17	0.97	0.53	0.53	0.53
7	Hanurav (1967)	Population in 1967	Population in 1957	20	0.30	0.30	0.97	0.50	0.50	0.51
8	Hanurav (1967)	Population in 1967	Population in 1957	19	0.45	0.44	0.97	0.47	0.47	0.50
9	Hanurav (1967)	Population in 1967	Population in 1957	16	0.66	0.65	0.99	0.47	0.47	0.49
10	Hanurav (1967)	Population in 1967	Population in 1957	17	0.51	0.52	0.96	0.48	0.48	0.50
11	Cochran (1963)	No.of persons per block	No. of rooms per block	10	0.15	0.14	0.65	0.53	0.53	0.54
12	Cochran (1963) Cities 1-16	No. of people in 1930	No. of people in 1920	16	0.98	0.98	0.99	0.44	0.43	0.50

Table 1 continued

13	Cochran (1963) Cities 17-32	No. of people in 1930	No. of people in 1920	16	1.14	1.19	0.98	0.39	0.38	0.40
14	Cochran (1963) Cities 33-49	No. of people in 1930	No. of people in 1920	17	0.79	0.91	0.97	0.46	0.45	0.49
15	Sukhatme (1954) Villages 1-10	No.of wheat acres in 1937	No. of wheat acres in 1936	10	0.65	0.59	0.98	0.46	0.45	0.49
16	Sukhatme (1954) Villages 11-20	No. of wheat acres in 1937	No. of wheat acres in 1936	10	0.94	0.93	0.99	0.41	0.39	0.48
17	Sampford (1962)	Oats acreage in 1957	Total acreage in 1947	35	0.71	0.71	0.83	0.49	0.49	0.50
18	Sukhatme (1954) Circles 1-20	Wheat acreage	No. of villages	20	0.63	0.50	0.59	0.48	0.48	0.50
19	Sukhatme (1954) Circles 21-40	Wheat acreage	No. of villages	20	0.61	0.46	0.76	0.48	0.48	0.50
20	Sukhatme (1954) Circles 81-89	Wheat acreage	No. of villages	9	0.47	0.65	0.69	0.45	0.44	0.47

3.1. Stabilities of the estimator.

We first consider the stabilities of the estimators. Table 2 gives the percent gains in efficiency of the estimators over Brewer's estimator (i.e., [V(Brewer's est.)/V(est.)-1]x100), for the populations of Table 1. The following tentative conclusions can be drawn from Table 2: (1) For the natural populations, the efficiencies of Hanurav's, Brewer's and Fellegi's estimators are essentially identical; for the artificial populations, however, Hanurav's estimator appears slightly less efficient than the latter. (2) Murthy's estimator is consistently more efficient than the R.H.C. estimator and the gains are considerable for the artificial populations and the natural populations with small N and moderately large C.V.(x) (Natural pops. 4 and 5). For the natural populations, the R.H.C. estimator compares favorably with the estimators of Brewer, Fellegi and Hanurav. (3) The loss in efficiency of Des Raj's estimator over Murthy's estimator is very small for the natural populations, excepting populations 4 and 5. It is, however, considerable for the artificial populations. (4) Lahiri's estimator is considerably more efficient than the others when one or two units in the population have large x_i relative the x_i of the remaining units and samples containing these units give good estimates of Y (e.g., natural pops. 12-14). However, it is considerably less efficient for other populations and, in fact, less efficient than the customary estimator in sampling with replacement for six of the natural populations. (5) For the natural populations, Murthy's estimator appears more efficient than those of Brewer, Fellegi and Hanurav (gains range from -2 to 18%). However, for the artificial populations it is not clear cut. In any case, it appears on the whole that Murthy's estimator compares favorably with those of Brewer, Fellegi and Hanurav.

3.2. Stabilities of the variance estimators.

We now compare the stabilities of the variance estimators. Table 3 gives the percent gains in efficiency of the variance estimators over Brewer's variance estimator (i.e., 100 x [C.V.^{2}(Brewer's variance estimator)/C.V.^{2}(variance estimator)-1]) for the populations of Table 1. The following tentative conclusions can be drawn from Table 3: (1) Lahiri's variance estimator is considerably less efficient than the others for natural as well as artificial populations. Henceforth, we shall exclude Lahiri's method from further discussion. (2) Stabilities of Murthy's and Des Raj's variance estimators are essentially equal. It is, however, not true that Murthy's variance estimator will always be more stable than the latter. (3) The R.H.C. variance estimator is more efficient than Murthy's and other variance estimators. (4) Murthy's variance estimator is consistently more efficient than those of Brewer, Fellegi and Hanurav. The gains are considerable for several of the artificial as well as natrual populations. (5) For the

Table 2. Percent gains in efficiency* of the estimators over Brewer's estimator.

Pop. No.	Hanurav	Fellegi	Murthy	Des Raj	R.H.C.	Lahiri	With Rep.
			Artificial	Populations			
1	4	1	- 8	-19	-23	-31	-39
2	-8	-2	5	- 2	- 3	1	-22
3	-3	-2	-10	-23	-15	-22	-44
4	1	3	-15	-32	-29	-41	-52
5	-1	-0	- 1	- 6	- 3	- 9	-22
6	-4	-1	5	-21	-19	-37	-46
7	-0	-1	61	32	40	19	- 7
			Natural	Populations			
1	+0	+0	- 1	- 1	- 2	-16	- 7
2	-0	-0	- 0	- 0	- 1	-11	- 6
3	-0	-0	1	+ 0	1	1	- 7
4	-3	-2	4	- 1	- 7	-31	-17
5	-0	+0	18	12	7	5	- 5
6	-0	-0	+ 0	- 1	+ 0	1	-11
7	-0	-0	+ 0	+ 0	+ 0	1	- 5
8	+0	+0	- 0	- 0	- 2	-13	- 7
9	+0	+0	- 0	- 1	- 3	-17	-10
10	+0	-0	- 0	- 1	- 2	-12	- 8
11	+0	+0	1	- 1	1	4	-10
12	+0	+0	6	5	4	34	- 3
13	+0	-0	9	8	7	511	- 1
14	-0	-0	4	4	3	33	- 3
15	1	+0	- 2	- 4	- 6	-17	-17
16	1	+0	7	4	3	15	- 9
17	-0	-0	+ 0	- 0	- 1	-17	- 4
18	-0	-0	1	+ 0	- 0	- 4	- 5
19	+0	+0	+ 0	+ 0	- 0	- 2	- 6
20	-1	-0	6	4	5	28	- 7

*+0 and -0 indicate that the actual values are positive and negative respectively.

natural populations, stabilities of Brewer's, Fellegi's and Hanurav's variance estimators are about equal, except that, for populations 12 and 16, the gains in efficiency of Hanurav's variance estimator over that of Brewer are 15% and 17% respectively. However, for these two populations Murthy's estimator performs considerably better. For the artificial populations, Hanurav's variance estimator appears less stable than that of Brewer. In this connection, it is interesting to note that Hanurav's variance estimator is less stable than Brewer's for the artificial population 6, although his ratio φ is considerably larger than Brewer's φ (see Table 1). This clearly shows that Hanurav's criterion does not always guarantee a more stable variance estimator – the stability also depends on the differences $y_i/x_i - y_j/x_j$. (6) For all populations, the R.H.C. variance estimator is more efficient than the customary variance estimator in sampling with replacement. This is, however, not true with regard to the other variance estimators. (7) Gains in efficiency of the R.H.C. variance estimator over Murthy's are not large, excepting few extreme cases.

4. Formulae under the super-population model.

Using the model (1) we get the following average variances of the estimators:

$$\epsilon V_1 = \frac{aX^g}{2} \sum_1^N (1-2p_i)p_i^{g-1} \tag{28}$$

$$\epsilon V_2 = \frac{aX^g}{2} \sum_{i<j}^N\sum p_i^{g-1}p_j(2-p_i-p_j) \tag{29}$$

$$\epsilon V_3 = 2aX^g \sum_{i<j}^N\sum p_i^{g-1}p_j \frac{1-p_i-p_j}{2-p_i-p_j} \tag{30}$$

$$\epsilon V_4 = ac_0c_1X^g \sum_1^N (1-p_i)p_i^{g-1}. \tag{31}$$

In view of our results in Section 3 we have not included Lahiri's method here, but it is known that $\epsilon V_5 \gtrless \epsilon V_1$ according as $g \gtrless 1$.

It is clear from (28) that all methods with $\pi_i=2p_i$ and using the Horvitz-Thompson estimator have the same average variance. It is also known

Table 3. Percent gains in efficiency of the variance estimators over Brewer's variance estimator.

Pop. No.	Hanurav	Fellegi	Murthy	Des Raj	R.H.C.	Lahiri	With Rep.
			Artificial	Populations			
1	- 5	2	61	57	53	- 99	-10
2	-16	-4	10	- 0	28	-100	-26
3	- 7	-5	31	14	74	-100	-16
4	8	8	418	440	512	- 99	164
5	- 2	-1	- 0	- 2	2	- 94	-30
6	-10	-5	185	252	789	-100	200
7	- 3	-8	2548	2755	6083	- 99	2363
			Natural	Populations			
1	1	+0	- 3	- 3	- 5	- 99	-13
2	- 0	-0	+ 0	- 0	- 1	- 89	- 7
3	- 0	-0	6	6	10	-100	- 6
4	- 2	2	38	24	59	- 99	8
5	- 5	7	301	303	508	- 97	322
6	- 0	-0	3	3	7	-100	-12
7	0	+0	1	1	2	-100	- 5
8	- 0	+0	1	1	1	-100	- 9
9	- 2	-0	4	4	5	-100	- 9
10	- 1	-0	4	4	9	-100	- 3
11	+ 0	+0	3	3	5	- 99	-12
12	15	2	22	21	39	-100	19
13	- 1	+2	39	34	54	26	30
14	2	+0	8	7	15	- 96	2
15	2	-0	13	11	20	-100	- 6
16	17	5	38	38	75	-100	36
17	1	+0	2	2	4	- 97	- 3
18	- 0	+0	4	4	8	- 90	- 3
19	- 0	+0	5	5	9	- 90	+ 0
20	- 1	1	16	13	27	- 87	3

that $\epsilon V_4 \gtrless \epsilon V_1$ according as $g \gtrless 1$ and $\epsilon V_4 = \epsilon V_1$ if $g=1$; $\epsilon V_3 < \epsilon V_4$ if $g=2$; $\epsilon V_1 < \epsilon V_3$ if $g=2$ and $\epsilon V_1 > \epsilon V_3$ if $g=1$.

For the comparison of variance estimators, we further assume that the e_i are normally distributed so that $\epsilon(e_i^4) = 3a^2x_i^{2g}$. The most appropriate measure of the stability of v_i appears to be $\epsilon[C.V.^2(v_i)]$, i.e., average $(C.V.)^2$ of the variance estimator. However, since $\epsilon[C.V.^2(v_i)]$ is the expectation of the ratio of two random variables, the evaluation is difficult. We have, therefore, used the alternative measure

$$\frac{\epsilon E[v_i - \epsilon V_i]^2}{(\epsilon V_i)^2} = \frac{\epsilon[EV_i^2] - (\epsilon V_i)^2}{(\epsilon V_i)^2} \tag{32}$$

which is readily evaluable. Notice that (32) actually measures the variability of v_i around the average variance ϵV_i. We, however, expect that (32) and $\epsilon[C.V.^2(v_i)]$ would lead to same conclusions.

To evaluate (32) we need the following formulae:

$$a^{-2}\epsilon[Ev_1^2] = \frac{3}{16} X^{2g} \sum_{i<j}^{N}\sum \frac{(4p_ip_j - \pi_{ij})^2}{\pi_{ij}} \cdot (p_i^{g-2}+p_j^{g-2})^2 \tag{33}$$

$$a^{-2}\epsilon[Ev_2^2] = \frac{3}{8} X^{2g} \sum_{i<j}^{N}\sum p_ip_j((1-p_i)^3 \cdot (p_i^{g-2}+p_j^{g-2})^2 \tag{34}$$

$$a^{-2}\epsilon[Ev_3^2] = 3X^{2g} \sum_{i<j}^{N}\sum p_ip_j \cdot \frac{(1-p_i)(1-p_j)(1-p_i-p_j)^2}{(2-p_i-p_j)^3} \cdot (p_i^{g-2}+p_j^{g-2})^2 \tag{35}$$

and

$$a^{-2}\epsilon[Ev_4^2] = c_0^2X^{2g}[3c_1(B_{20}^2-4B_{41}) + 3c_2(4B_{02}B_{42}-B_{02}B_{21}^2+2B_{21}B_{2,-1}$$

Table 4. Percent gains in average efficiency of the estimators over Brewer's estimator (under the super-population model for g=1.0, 1.5, 1.75, 2.0)

	g=1		g=1.5			g=1.75			g=2.0		
Pop. No.	Murthy	Des Raj	Murthy	Des Raj	R.H.C.	Murthy	Des Raj	R.H.C.	Murthy	Des Raj	R.H.C.
				Artificial Populations							
1,2	4	-4	1	- 8	-11	+0	- 9	-13	-1	-11	-14
3,4	6	-9	2	-15	- 7	-1	-17	-11	-3	-20	-14
5	1	-4	+0	- 4	- 1	+0	- 4	- 1	-0	- 5	- 2
				Natural Populations							
1	+ 0	+0	+0	- 0	- 0	+0	- 0	- 1	-0	- 0	- 1
2	+ 0	+0	+0	- 0	- 0	+0	- 0	- 1	-0	- 0	- 1
3	1	+0	+0	- 0	- 1	+0	- 1	- 1	-0	- 1	- 2
4	10	5	4	- 2	-10	1	- 6	-15	-2	-10	-19
5	12	6	5	- 2	-11	-0	- 8	-17	-6	-14	-23
6	+ 0	-1	+0	- 1	- 0	+0	- 1	- 0	-0	- 1	- 0
7	+ 0	-0	+0	- 0	- 0	+0	- 0	- 0	-0	- 0	- 1
8	1	+0	+0	- 0	- 1	+0	- 0	- 1	-0	- 0	- 1
9	2	1	1	- 0	- 1	+0	- 1	- 2	-0	- 1	- 3
10	1	+0	+0	- 0	- 1	+0	- 0	- 2	-0	- 1	- 2
11	+ 0	-1	+0	- 1	- 0	+0	- 1	- 0	-0	- 1	- 0
12	4	2	2	+ 0	- 3	1	- 1	- 5	-1	- 3	- 7
13	6	4	3	1	- 4	1	- 2	- 7	-2	- 5	-11
14	3	2	1	+ 0	- 3	1	- 1	- 5	-0	- 2	- 6
15	2	+0	1	- 1	- 2	+0	- 2	- 3	-0	- 3	- 5
16	6	3	2	- 2	- 7	1	- 4	-10	-1	- 6	-12
17	1	1	+0	+ 0	- 1	+0	- 0	- 1	-0	- 0	- 2
18	1	+0	+0	- 0	- 1	+0	- 0	- 1	-0	- 1	- 1
19	1	+0	+0	- 0	- 1	+0	- 0	- 1	-0	- 0	- 1
20	3	+0	1	- 2	- 5	+0	- 3	- 6	-1	- 4	- 8

$$- 2B_{21}B_{20}+B_{21}^2+B_{43}-3B_{42}+13B_{41}-8B_{40}+B_{0,-3}B_{43}$$
$$- 2B_{0,-2}B_{43})+3c_3(B_{0,-2}B_{21}^2-4B_{0,-2}B_{42}$$
$$- 2B_{20}^2-4B_{21}B_{2,-1}+4B_{21}B_{20}-B_{21}^2-B_{43}+4B_{42}$$
$$- 10B_{41}+12B_{40}-2B_{0,-3}B_{43}+3B_{0,-2}B_{43})] \quad (35)$$

where $c_0,\ldots,c_3$ are as before and

$$B_{0j} = \sum_1^N p_t^{-j},\ B_{2j} = \sum_1^N p_t^{g-j}$$

and

$$B_{4j} = \sum_1^N p_t^{2g-j}. \quad (36)$$

5. Empirical results under the super-population model.

5.1. Stabilities of the estimators.

Table 4 gives the percent gains in average efficiency of the estimators over Brewer's estimator (i.e., 100 x [ϵV(Brewer's est.)/ϵV(est.)-1]) for the populations of Table 1 (excluding artificial populations 6 and 7) and g=1.0, 1.5, 1.75 and 2.0. The following tentative conclusions can be drawn from Table 4: (1) Murthy's estimator is more efficient than the Horvitz-Thompson estimator (i.e., Brewer's, Fellegi's and Hanurav's) for $g \leq 1.75$; however, the gains are small for $g \geq 1.5$. Moreover, the losses in efficiency over the latter for g=2 are small. (2) Murthy's estimator is consistently more efficient than the R.H.C. estimator and the gains are considerable for several populations. (3) Gains in efficiency of Murthy's estimator over Des Raj's are considerable for populations with small N or moderately large C.V.(x). (4) Des Raj's estimator is less efficient than the Horvitz-Thompson estimator for $g \geq 1.5$.

5.2. Stabilities of the variance estimators.

Using the measure (32) we have computed the percent gains in average efficiency of the variance estimators over Brewer's for g=1.0, 1.5, 1.75 and 2.0, and the results are given in Table 5. The following tentative conclusions can be drawn from Table 5: (1) As before, the stabilities of Murthy's and Des Raj's variance estimators are essentially equal. (2) The R.H.C. variance estimator is consistently more efficient than Murthy's and other variance estimators. However, as before, the gains over Murthy's variance estimator are not large, excepting for few extreme cases. (3) Murthy's variance estimator is consistently

Table 5. Percent gains in average efficiency of the variance estimators over Brewer's variance estimator (under the assumption of a super-population model for g = 1.0, 1.5, 1.75, 2.0).

	g=1.0					g=1.5				
Pop. No.	Murthy	Des Raj	R.H.C.	Hanurav	Fellegi	Murthy	Des Raj	R.H.C.	Hanurav	Fellegi
				Artificial Populations						
1,2	50	51	82	15	5	43	43	62	8	4
3,4	155	165	247	8	9	157	162	219	6	6
5	8	9	13	+ 0	1	6	6	9	+ 0	+0
				Natural Populations						
1	3	3	5	+ 0	+0	2	2	3	+ 0	+0
2	3	3	6	+ 0	+0	2	2	3	+ 0	+0
3	6	6	10	1	+0	4	4	6	+ 0	+0
4	70	67	130	15	9	87	86	143	12	8
5	277	268	433	- 4	8	370	362	543	- 3	4
6	2	2	3	+ 0	+0	1	1	2	+ 0	+0
7	2	2	3	+ 0	+0	1	1	2	+ 0	+0
8	8	8	16	+ 0	+0	2	2	4	+ 0	+0
9	10	10	18	2	+0	8	8	13	1	+0
10	8	8	16	1	+0	4	4	7	+ 0	+0
11	1	1	2	+ 0	+0	1	1	1	- 0	+0
12	24	23	42	13	2	22	22	36	8	1
13	54	52	85	- 0	2	68	65	97	+ 0	1
14	16	16	30	7	1	14	14	24	4	1
15	17	17	31	3	1	13	13	21	1	+0
16	34	33	70	9	3	35	34	60	7	3
17	4	4	8	1	+0	3	3	5	+ 0	+0
18	5	5	9	0	+0	3	3	5	+ 0	+0
19	4	4	8	0	+0	2	2	4	+ 0	+0
20	26	26	45	3	1	22	22	34	1	1

more efficient than those of Brewer, Fellegi and Hanurav and the gains are considerable for several of the artificial as well as the natural populations. (4) Fellegi's variance estimator is consistently more efficient than Brewer's; however, the gains are small. The efficiencies of Hanurav's and Fellegi's variance estimators are essentially equal for $g \geq 1.75$ although the latter is consistently more efficient for g=2. Hanurav's variance estimator is slightly more efficient for $g \leq 1.5$.

6. Concluding Remarks.

It appears that our results under the super-population model are in agreement with those from the empirical study using the actual y-data. The following major conclusions may be drawn from our studies: (1) Murthy's method is preferable over the other methods when a stable estimator as well as a stable variance estimator are required. (2) The R.H.C. variance estimator is the most stable, but the R.H.C. estimator might lead to significant losses in efficiency. (3) Hanurav's method does not lead to significant improvements over Fellegi's or Brewer's methods with regard to stability of the variance estimator.

It should be noted that, for the case of $n > 2$, some of these methods are either not applicable (e.g., Brewer's method) or become computationally cumbersome (e.g., Murthy's method when n is moderate). Therefore, the case of $n > 2$ could lead to completely different conclusions. A detailed investigation of the stabilities for $n > 2$ is underway and the results will be reported in a subsequent paper.

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References.

Brewer, K. R. W. (1963), "A model of systematic sampling with unequal probabilities", Australian Journal of Statistics, 5, 5-13.

Carroll, J. L. and Hartley, H. O. (1964), "The symmetric method of unequal probability sampling without replacement", unpublished to date; abstract in Biometrics, (1964), 20, 908-9.

Cochran, W. G. (1963), Sampling Techniques, 2nd Ed., New York: John Wiley and Sons.

Des Raj (1956), "Some estimators in sampling with varying probabilities without replacement", Journal of the American Statistical Association, 51, 269-84.

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Table 5 continued

Pop. No.	g=1.75 Murthy	Des Raj	R.H.C.	Hanurav	Fellegi	g=2.00 Murthy	Des Raj	R.H.C.	Hanurav	Fellegi
				Artificial Populations						
1,2	38	37	49	5	3	32	30	36	1	2
3,4	152	153	195	4	5	144	141	170	3	3
5	5	5	6	-0	+0	4	3	4	-0	+0
				Natural Populations						
1	1	1	2	+0	+0	1	+ 0	1	-0	+0
2	1	1	2	+0	+0	1	+ 0	1	-0	+0
3	3	3	4	+0	+0	2	1	2	-0	+0
4	82	79	112	7	5	70	63	76	3	3
5	400	377	518	-0	2	406	361	457	-1	+0
6	1	1	1	-0	+0	+ 0	+ 0	1	-0	+0
7	1	1	1	-0	+0	+ 0	+ 0	+ 0	-0	+0
8	1	1	1	-0	+0	+ 0	+ 0	+ 0	-0	+0
9	6	6	9	+0	+0	4	3	5	-0	+0
10	2	2	4	+0	+0	1	1	1	-0	+0
11	+ 0	+ 0	1	-0	+0	+ 0	+ 0	+ 0	-0	+0
12	19	18	27	4	1	13	11	15	-2	+0
13	66	61	86	+0	1	58	51	66	-0	+0
14	11	10	17	1	+0	7	5	8	-1	+0
15	10	9	14	1	+0	7	6	8	-0	+0
16	28	26	40	4	2	19	17	22	+0	1
17	2	2	3	+0	+0	+ 0	+ 0	1	-0	+0
18	2	2	3	+0	+0	1	1	1	-0	+0
19	2	1	3	+0	+0	1	1	1	-0	+0
20	18	17	25	+0	1	13	12	15	-0	+0

Des Raj (1965), "Variance estimation in randomized systematic sampling with probability proportional to size", Journal of the American Statistical Association, 60, 278-84.

Durbin, J. (1967), "Estimation of sampling errors in multistage surveys", Applied Statistics, 16, 152-64.

Fellegi, I. P. (1963), "Sampling with varying probabilities without replacement: rotating and non-rotating samples", Journal of the American Statistical Association, 58, 183-201.

Godambe, V. P. (1966), "A new approach to sampling from a finite population-I", Journal of the Royal Statistical Society, Series B, 28, 310-19.

Hájek, J. (1964), "Asymptotic theory of rejective sampling with varying probabilities from a finite population", Annals of Mathematical Statistics, 35, 1491-523.

Hanurav, T. V. (1967), "Optimum utilization of auxiliary information: πPS sampling of two units from a stratum", Journal of the Royal Statistical Society, Series B, 29, 374-91.

Hartley, H. O. and Rao, J. N. K. (1962), "Sampling with unequal probabilities and without replacement", Annals of Mathematical Statistics, 33, 350-74.

Hartley, H. O. (1966), "Systematic sampling with unequal probability and without replacement", Journal of the American Statistical Association, 61, 739-48.

Horvitz, D. G. and Thompson, D. J. (1952), "A generalization of sampling without replacement from a finite universe", Journal of the American Statistical Association, 47, 663-85.

Kish, L. (1965), Survey Sampling. New York: John Wiley and Sons.

Lahiri, D. B. (1951), "A method for sample selection providing unbiased ratio estimates", International Statistical Institute Bulletin, 33, No. 2, 133-40.

Madow, W. G. (1949), "On the theory of systematic sampling II", Annals of Mathematical Statistics, 20, 333-54.

Murthy, M. N. (1957), "Ordered and unordered estimators in sampling without replacement", Sankhyā, 18, 379-90.

Rao, J. N. K., Hartley, H. O., and Cochran, W. G. (1962), "On a simple procedure of unequal probability sampling without replacement", Journal of the Royal Statistical Society, Series B, 24, 482-91.

Rao, J. N. K. (1963), "On three procedures of unequal probability sampling without replacement", Journal of the American Statistical Association, 58, 202-15.

Rao, J. N. K. (1965), "On two simple schemes of unequal probability sampling without replacement", Journal of the Indian Statistical Association, 3, 173-80.

Rao, J. N. K. (1966), "On the relative efficiency of some estimators in PPS sampling for multiple characteristics", Sankhyā, 28, 61-70.

Sampford, M. R. (1962). An introduction to sampling theory. Edinburgh and London: Oliver and Boyd Ltd.

Stevens, W. L. (1958), "Sampling without replacement with probability proportional to size", Journal of the Royal Statistical Society, Series B, 20, 393-7.

Sukhatme, P. V. (1954). Sampling theory of surveys with applications. Ames, Iowa: Iowa State College Press.

Yates, F. and Grundy, P. M. (1953), "Selection without replacement from within strata with probability proportional to size", Journal of the Royal Statistical Society, Series B, 15, 253-61.